expansion and will not be affected by elastic strains. However, for the case of the clamped shell, neglect of transverse normal elastic strain will cause significant error in computing edge stresses. Thus, it would appear that for shells made of materials having high transverse Poisson's ratio (ν_{13}), high thermal expansion coefficient ratios (α_{33}/α_{11}) and subjected to edge restraints, normal elastic, and thermal stresses are significant and must be accounted for in stress computations.

For those cases when elastic normal stress can be ignored in comparison with transverse normal thermal stress, a closed-form solution for both single and laminated shells[5] is possible for the static case.

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Mode Shapes and Frequencies of Clamped-Clamped Cylindrical Shells

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Nomenclature

a = mean radius of shell
E = modulus of elasticity
h = thickness of shell wall
k = $h^2/12a^2$
L = length of shell
m = number of axial half-waves
n = number of circumferential waves
t = time
u, v, w = middle surface axial, circumferential, and radial displacements
x, ϕ = axial and circumferential coordinates of shell middle surface
ν = Poisson's ratio
ρ = mass density of shell material
ω = circular frequency
Ω = frequency parameter, $\omega a[\rho(1-\nu^2)/E]^{1/2}$

Introduction

IN a recent investigation into the influence of a blast environment on the dynamic response of a missile structure it became appropriate to examine the vibration characteristics of a finite length, thin cylindrical shell that was rigidly clamped at both ends. Although the shell vibration problem had previously been studied by several investigators, including Forsberg,[1,2] Smith and Haft,[3] and Vronay and Smith,[4] it became important in our case to develop a simple method for determining the shell's modal response to a uniform radial blast load. In the subsequent analysis it was noted that a misconception probably has arisen in previous comparisons between axisymmetric mode shapes with $n = 0$ and mode shapes with $n \geqq 1$.

An exact solution to Flügge's[5] thin shell equations was obtained using the approach suggested in Ref. 4, which eliminates the usual difficulty of deriving a new set of real solutions each time there is a change in the form of the roots of the shell's equations of motion.[1–3,6] This difficulty is overcome simply by working with an unaltered set of complex modal functions throughout the solution process. The process is carried out one step further in that the final set of mode shapes is taken directly from the real component of these complex modal functions.

Natural frequencies and mode shapes are presented here for the first six circumferential wave numbers ($n = 0$–5) and the two lowest radial modes of the shell model studied in Refs. 3 and 4. The special case of axisymmetric vibration ($n = 0$) is treated separately, but the case of beam-type vibrations ($n = 1$) is treated along with the higher wave numbers.

Solution of Shell Equations

In terms of the component displacements u, v, and w the general modal solution to Flügge's[5] differential equations of motion for a finite length, thin, circular, cylindrical shell can be written as

$$\begin{aligned} u &= \sum_i A_i e^{\lambda_i x/a} \cos n\phi \sin \omega t \\ v &= \sum_i B_i e^{\lambda_i x/a} \sin n\phi \sin \omega t \\ w &= \sum_i C_i e^{\lambda_i x/a} \cos n\phi \sin \omega t \end{aligned} \tag{1}$$

Using these mode shapes, the differential equations lead to two types of polynomial expansions in λ_i. For the axisymmetric case with $n = 0$ we obtain the sixth-order equation†

$$\lambda_i^6 + D_3\lambda_i^4 + D_2\lambda_i^2 + D_1 = 0 \tag{2}$$

but for all other cases, we get the eighth-order equation

$$\lambda_i^8 + D_4\lambda_i^6 + D_3\lambda_i^4 + D_2\lambda_i^2 + D_1 = 0 \tag{3}$$

where the coefficients D are functions of n, Ω, ν, and k and each λ_i is one of either a real or complex pair of roots.

The six roots of Eq. (2) ($i = 1$–6) and the eight roots of Eq. (3) ($i = 1$–8) can be used to solve for the constants A_i and B_i in terms of the C_i's by applying the homogeneous restrictions of the equations of motion (for the axisymmetric case $v = B_i = 0$). Typically each coefficient relationship will either be real or complex, so that the component modal displacements in Eq. (1) will be expressed, as in Ref. 4, as a sum of complex quantities.

We assume that both ends of the shell are clamped and axially constrained so that the eight boundary conditions are

$$u = v = w = \partial w/\partial x = 0 \quad \text{at} \quad x = 0 \quad \text{and} \quad x = L \tag{4}$$

For the axisymmetric case, v is identically zero for all values of x, so that only six of these boundary conditions need be satisfied for $n = 0$.

Application of Eq. (1) to the boundary conditions leads to a set of homogeneous equations in terms of the unknown constants C_i

$$[Z]\{C_i\} = \{0\} \tag{5}$$

where $i = 1$–6 for $n = 0$; and $i = 1$–8 for $n \geqq 1$. The elements of the Z matrix are principally related to the frequency parameter Ω through either Eqs. (2) or (3). Since Eq. (5) is homogeneous, the remaining problem is to calculate the values of Ω that eventually will make the determinant $|Z|$ vanish.

The correct values Ω_j are essentially eigenvalues that determine the natural frequencies of the shell. The eigenvector constants $\{C_i\}_j$ corresponding to each Ω_j determine the shell's orthogonal mode shapes through Eq. (1). Whichever method is

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† For $n = 0$, a separate second-order solution also exists that involves only pure torsional motion.[2]

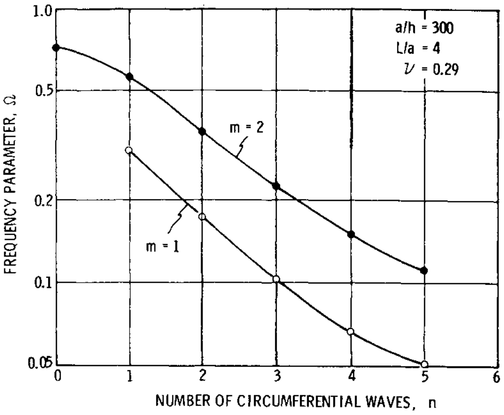

Fig. 1 Frequency distribution for clamped steel shell.

used to compute these constants,[7] usually the constants will form mode shapes that are complex functions. It follows from the nature of the solution that, in the complex plane, all these initial mode shape functions lie on a straight line that passes through the origin. Thus each complex quantity is related to its real component by an identical constant of proportionality, and, since we are accustomed to working with real displacements, it is convenient (and valid) to use these proportional real components (normalized as desired) as the shell's orthogonal mode shapes.

Results and Conclusions

Natural frequencies for the first six circumferential wave numbers and the two lowest radial modes are presented in Table 1 for a clamped-clamped steel shell with $L = 12$ in., $a = 3$ in., and $h = 0.01$ in. Comparisons are made in Table 1 with theoretical results for $n \geqq 1$ that were obtained for the same shell in Refs. 3 and 4. These earlier results are from solutions to Flügge's equations that were computed only after using the uncoupling conditions suggested by Yu.[8] The variation in the frequency parameter for this case is illustrated in Fig. 1. Of particular interest is the spatial phase shift in Fig. 2 between the lowest frequency radial modes for $n = 0$ and for $n \geqq 1$.

The technical significance of this shift lies in the fact that for $n = 0$, the lowest frequency radial mode actually is antisymmetric ($m = 2$) and thus cannot respond to a uniform radial blast load. This important result agrees with Ref. 6 but is in contrast to a misconception which might easily arise from earlier work and in particular the monograph of Ref. 9. The misconception is that the lowest mode of a clamped shell with $n = 0$ is symmetric ($m = 1$). Although the correct situation was noted by Forsberg[2] in his analysis of the axisymmetric vibration of a clamped shell the proper interpretation that $m = 2$ is lacking in recent publications such as Ref. 9.

Table 1 Frequencies (Hz) for clamped shell[a]

m	Source	n = 0	1	2	3	4	5
1	Present		3425	1917	1153	762	575
	Ref. 3		3423	1917	1154	765	581
	Ref. 4		3427	1918	1145	765	580
2	Present	8017	6412	3903	2536	1751	1284
	Ref. 3		6412	3903	2536	1752	1287
	Ref. 4		6423	3905	2538	1753	1287

[a] $L = 12.0$ in., $a = 3.0$ in., $h = 0.01$ in., $E = 29.6 \times 10^6$ psi, $\nu = 0.29$, $\rho = 0.733 \times 10^{-3}$ lb.-sec^2/in.^{4}

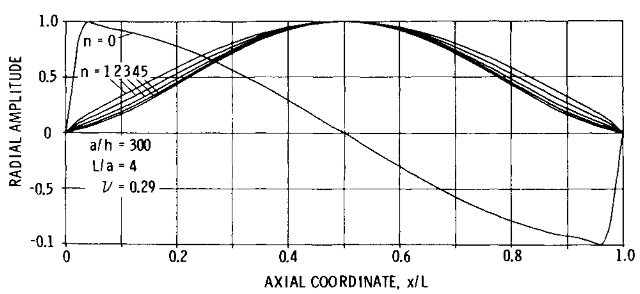

Fig. 2 Initial radial mode shape for clamped steel shell.

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Nonlinear Creep Analysis by Assumed Stress Finite Element Methods

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I. Introduction

FINITE element methods for analyzing the creep behavior of structures have been introduced by many authors.[1–8] They are invariably incremental elastic solutions by direct stiffness methods derived by the conventional assumed displacement approach and based on equivalent nodal forces due to the creep strains that are anticipated during each time increment. When the incremental solutions are carried out without iteration, they are analogous to the Euler's method for numerical integration of differential equations and hence require very small time increments. Indeed, numerical instability for such analysis has been experienced by some investigators when the time increments are not kept sufficiently small.[5] In many available computing codes, the incremental solutions are accompanied by iterative schemes.[7]

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